Letters

General Form of Green's Function for Multilayer Microstrip Line with Rectangular Side Walls

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***Abstract*—This letter shows the general form of Green's functions for multilayer microstrip lines with rectangular side walls derived for the eight cases classified according to the boundary conditions by taking account of the reciprocity relation for Green's function and using the method of separation of variables.**

The TEM wave approximation is well known to be reasonable in many cases when the cross-sectional dimensions of a transmission line with a nonuniform medium are much smaller than the wavelength to be used. Therefore, the calculation of characteristic impedances, phase velocities, and attenuation constants of various transmission lines based on the TEM wave approximation is useful for the design of microwave integrated-circuit structures. These parameters often can be calculated using Green's functions [1]–[6]; however, the Green's function satisfying the boundary conditions must be obtained first. The use of Green's functions converts a differential equation together with the boundary conditions to an integral equation, which in many cases is more readily attacked by approximate techniques. Therefore, to find the Green's function satisfying boundary conditions is an important problem for calculating the parameters by using the Green's function. Although the general form of Green's functions for a multilayer microstrip line with rectangular side walls can be easily derived by extending the method described in [2]–[4], it has not been obtained in the literature; the reason seems to be that the strip structures rarely have more than three dielectric layers from a practical point of view.

The purpose of this letter, from mathematical interests and the usefulness in programming to calculate the parameters, is to show the general form of Green's functions for such a line derived by taking account of the reciprocity relation [7] and extending the method described in [3], [4], and [8, pp. 52–54].

We consider the region R in a two-dimensional space shown in Fig. 1 for determining the Green's function of an n-layer microstrip line with rectangular side walls. The region R is composed of the regions R_i, where ($i = 1,2,\cdots,n$). Let the region R_i be filled with a homogeneous dielectric of permittivity ε_i. This problem can be classified in the eight cases according to the boundary conditions; that is, the Dirichlet and mixed conditions, as illustrated in Fig. 2, in which the solid lines on outer boundaries indicate electric walls $G = 0$, and the broken lines indicate magnetic walls $\partial G/\partial \nu = 0$, where ν is the external normal to the walls. From a practical point of view, cases 3, 7, and 8 in Fig. 2 are interesting cases: for example, case 3 was always studied for $n = 2$ by Gish and Graham [2], case 7 for $n = 2$ by Allen [5], and case 8 for $n = 3$ by Yamashita and Atsuki [3], [4], and for $n = 4$ by Albrey and Gunn [6]. The Green's function for the eight cases can be derived by taking account of the reciprocity relation [7, eq. (14)] and extending the

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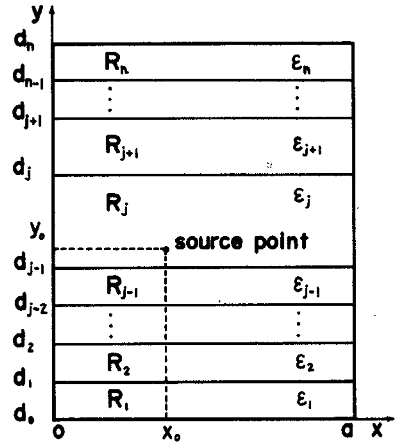

Fig. 1. Illustration for determining the Green's function in a rectangular region R with n-dielectric layers.

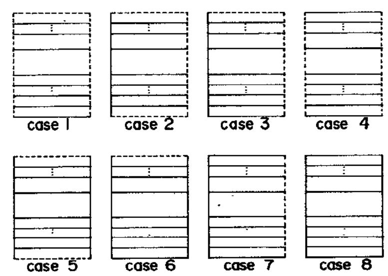

Fig. 2. Classification of Fig. 1 based on the boundary conditions.

method described in [3], [4], and [8, pp. 52–54]. For example, the Green's function for case 1 can be obtained by substituting the boundary conditions at the interfaces $y = d_t$, where ($t = 1, 2,\cdots, n - 1$) into the following equations, satisfying the Poisson's equation and the outer boundary conditions when letting the Green's function $G(x,y;x_0,y_0)$ at an observation point (x,y) in R_i due to the unit charge source at a point (x_0,y_0) in R_j denote by $G(i)$

$$G(1) = \frac{2}{a\varepsilon_j}\{\tfrac{1}{2}C_{1,0}(y - d_0) + \Sigma' C_{1,m}\, S_m(j,x,x_0)SH_m'(j,y_0)\sinh \gamma_m(y - d_0)\} \tag{1a}$$

$$G(i) = \frac{2}{a\varepsilon_j}\{\tfrac{1}{2}C_{i,0}(y + C_{i,0}') + \Sigma' C_{i,m}S_m(j,x,x_0)SH_m'(j,y_0)SH_m'(i,y)\}, \qquad i = 2, 3,\cdots, j - 1 \tag{1b}$$

$$G(j) = \frac{2}{a\varepsilon_j}\begin{cases}\tfrac{1}{2}(y + C_{j,0}') + \Sigma' S_m(j,x,x_0)SH_m'(j,y_0)SH_m(j,y), & d_{j-1} \le y \le y_0\\ \tfrac{1}{2}(y_0 + C_{j,0}') + \Sigma' S_m(j,x_0,x)SH_m'(j,y)SH_m(j,y_0), & y_0 \le y \le d_j\end{cases} \tag{1c}$$

$$G(k) = \frac{2}{a\varepsilon_j}\{\tfrac{1}{2}(y_0 + C_{j,0}')$$
$$+ \Sigma' C_{k,m} S_m(j,x_0,x) SH_m'(k,y) SH_m(j,y_0)\},$$
$$k = j+1, j+2, \cdots, n-1 \quad (1d)$$

$$G(n) = \frac{2}{a\varepsilon_j}\{\tfrac{1}{2}(y_0 + C_{j,0}')$$
$$+ \Sigma' C_{n,m} S_m(j,x_0,x) SH_m(j,y_0) \cosh \gamma_m(d_n - y)\} \quad (1e)$$

where

$$\Sigma' \equiv \sum_{m=1,2,3}^{\infty}, \qquad \gamma_m = \frac{m\pi}{a}$$

$$S_m(j,x,x_0) \equiv \frac{\cos \gamma_m x \cos \gamma_m x_0}{\gamma_m (C_{j,m}' - C_{j,m})}$$

$$SH_m(j,y) \equiv \sinh \gamma_m y + C_{j,m} \cosh \gamma_m y$$

$$SH_m'(i,y) \equiv \sinh \gamma_m y + C_{i,m}' \cosh \gamma_m y,$$
$$i = 2, 3, \cdots, n-1.$$

For example, the coefficients $C_{i,m} (m \neq 0)$ become as follows:

$$C_{i,m} = \frac{\prod_{t=i+1}^{j} \varepsilon_t^* YK_1(\varepsilon_{i-1}^*, \varepsilon_i^*, 0, d_{i-1}, \gamma_m)}{YK_1(\varepsilon_{j-1}^*, \varepsilon_j^*, 0, d_{j-1}, \gamma_m)},$$
$$i = 2, 3, \cdots, j-1. \quad (2)$$

We can derive the Green's functions for cases 2–8 similarly. Then, the general form of Green's functions for the eight cases in Fig. 2 can be expressed by the following equations:

$$G(i) = \frac{2}{a\varepsilon_0}\left\{\left(\alpha \prod_{t=i+1}^{j} \varepsilon_t^* + \beta\right) \sum_{m=0}^{0} + \prod_{t=i+1}^{j} \varepsilon_t^* \Sigma\right\}$$
$$\cdot \frac{A_m(x) A_m(x_0) B_m(j,y_0) C_m(i,y)}{\varepsilon_j^* \gamma_m D_m(1,d_0)},$$
$$i = 1, 2, \cdots, j-1 \quad (3a)$$

$$G(j) = \frac{2}{a\varepsilon_0}\begin{cases} \left(\xi \sum_{m=0}^{0} + \Sigma\right) \dfrac{A_m(x) A_m(x_0) B_m(j,y_0) C_m(j,y)}{\varepsilon_j^* \gamma_m D_m(1,d_0)}, \\ \qquad d_{j-1} \leq y \leq y_0 \\ \left(\xi \sum_{m=0}^{0} + \Sigma\right) \dfrac{A_m(x_0) A_m(x) B_m(j,y) C_m(j,y_0)}{\varepsilon_j^* \gamma_m D_m(1,d_0)}, \\ \qquad j = 1,2,\cdots,n, \; y_0 \leq y \leq d_j \end{cases} \quad (3b)$$

$$G(k) = \frac{2}{a\varepsilon_0}\left\{\left(\alpha \prod_{t=j}^{k-1} \varepsilon_t^* + \beta\right) \sum_{m=0}^{0} + \prod_{t=j}^{k-1} \varepsilon_t^* \Sigma\right\}$$
$$\cdot \frac{A_m(x_0) A_m(x) B_m(k,y) C_m(j,y_0)}{\varepsilon_j^* \gamma_m D_m(1,d_0)},$$
$$k = j+1, j+2, \cdots, n \quad (3c)$$

where ε^* = relative dielectric constant.

a) For $m \neq 0$

$$\Sigma \equiv \sum_{m=1,2,3}^{\infty} \text{ or } \sum_{m=1,3,5}^{\infty}, \qquad \gamma_m = \frac{m\pi}{a} \text{ or } \frac{m\pi}{2a} \quad (4)$$

$$A_m(x) \equiv \sin \gamma_m x \quad \text{or} \quad \cos \gamma_m x \quad (5)$$

$$B_m(i,y) \equiv MK_1(\varepsilon_i^*, \varepsilon_{i+1}^*, d_i, y, \gamma_m),$$
$$i = j, j+1, \cdots, n \quad (6)$$

$$C_m(i,y) \equiv YK_2(\varepsilon_{i-1}^*, \varepsilon_i^*, y, d_{i-1}, \gamma_m),$$
$$i = 1,2,\cdots,j \quad (7)$$

$$D_m(1,d_0) \equiv \begin{cases} MK_1(\varepsilon_1^*, \varepsilon_2^*, d_1, d_0, \gamma_m) \\ MK_2(\varepsilon_1^*, \varepsilon_2^*, d_1, d_0, \gamma_m) \end{cases} \quad (8)$$

$$MK_1(\varepsilon_i^*, \varepsilon_{i+1}^*, d_i, d_{i-1}, \gamma_m)$$
$$\equiv \varepsilon_i^* \cosh \gamma_m(d_i - d_{i-1}) MK_1(\varepsilon_{i+1}^*, \varepsilon_{i+2}^*, d_{i+1}, d_i, \gamma_m)$$
$$+ \varepsilon_{i+1}^* \sinh \gamma_m(d_i - d_{i-1}) MK_2(\varepsilon_{i+1}^*, \varepsilon_{i+2}^*, d_{i+1}, d_i, \gamma_m),$$
$$i = 1, 2, \cdots, n-1 \quad (9)$$

$$MK_2(\varepsilon_i^*, \varepsilon_{i+1}^*, d_i, d_{i-1}, \gamma_m)$$
$$\equiv \varepsilon_i^* \sinh \gamma_m(d_i - d_{i-1}) MK_1(\varepsilon_{i+1}^*, \varepsilon_{i+2}^*, d_{i+1}, d_i, \gamma_m)$$
$$+ \varepsilon_{i+1}^* \cosh \gamma_m(d_i - d_{i-1}) MK_2(\varepsilon_{i+1}^*, \varepsilon_{i+2}^*, d_{i+1}, d_i, \gamma_m),$$
$$i = 1, 2, \cdots, n-1 \quad (10)$$

$$YK_1(\varepsilon_i^*, \varepsilon_{i+1}^*, d_{i+1}, d_i, \gamma_m)$$
$$\equiv \varepsilon_i^* \cosh \gamma_m(d_{i+1} - d_i) YK_1(\varepsilon_{i-1}^*, \varepsilon_i^*, d_i, d_{i-1}, \gamma_m)$$
$$+ \varepsilon_{i+1}^* \sinh \gamma_m(d_{i+1} - d_i) YK_2(\varepsilon_{i-1}^*, \varepsilon_i^*, d_i, d_{i-1}, \gamma_m),$$
$$i = 1, 2, \cdots, n-1 \quad (11)$$

$$YK_2(\varepsilon_i^*, \varepsilon_{i+1}^*, d_{i+1}, d_i, \gamma_m)$$
$$\equiv \varepsilon_i^* \sinh \gamma_m(d_{i+1} - d_i) YK_1(\varepsilon_{i-1}^*, \varepsilon_i^*, d_i, d_{i-1}, \gamma_m)$$
$$+ \varepsilon_{i+1}^* \cosh \gamma_m(d_{i+1} - d_i) YK_2(\varepsilon_{i-1}^*, \varepsilon_i^*, d_i, d_{i-1}, \gamma_m),$$
$$i = 1, 2, \cdots, n-1 \quad (12)$$

$$MK_1(\varepsilon_n^*, \varepsilon_{n+1}^*, d_n, d_{n-1}, \gamma_m)$$
$$\equiv \cosh \gamma_m(d_n - d_{n-1}) \text{ or } \sinh \gamma_m(d_n - d_{n-1}) \quad (13)$$

$$MK_2(\varepsilon_n^*, \varepsilon_{n+1}^*, d_n, d_{n-1}, \gamma_m)$$
$$\equiv \sinh \gamma_m(d_n - d_{n-1}) \text{ or } \cosh \gamma_m(d_n - d_{n-1}) \quad (14)$$

$$YK_1(\varepsilon_0^*, \varepsilon_1^*, d_1, d_0, \gamma_m)$$
$$\equiv \cosh \gamma_m(d_1 - d_0) \text{ or } \sinh \gamma_m(d_1 - d_0) \quad (15)$$

$$YK_2(\varepsilon_0^*, \varepsilon_1^*, d_1, d_0, \gamma_m)$$
$$\equiv \sinh \gamma_m(d_1 - d_0) \text{ or } \cosh \gamma_m(d_1 - d_0). \quad (16)$$

b) For $m = 0$

$$\gamma_0 = 1 \qquad A_0(x) \equiv 1 \quad (17)$$

$$\alpha = \beta = \xi = 0 \text{ for cases 2, 3, 5, 6, 7, and 8} \quad (18)$$

$$\alpha = 0 \qquad \beta = \xi = 1 \qquad B_0(i,y) \equiv 1 \qquad D_0(1,d_0) \equiv 1$$

$$C_0(i,y) \equiv \frac{\varepsilon_j^* YK_4(\varepsilon_{i-1}^*, \varepsilon_i^*, y, d_{i-1})}{2 \prod_{t=1}^{i} \varepsilon_t^*},$$
$$i = 1,2,\cdots,n \text{ for case 1} \quad (19)$$

$$\alpha = \xi = 1 \quad \beta = 0 \qquad B_0(i,y) \equiv MK_3(\varepsilon_i^*, \varepsilon_{i+1}^*, d_i, y)$$

$$D_0(1,d_0) \equiv MK_3(\varepsilon_1^*, \varepsilon_2^*, d_1, d_0),$$

$$C_0(i,y) \equiv YK_4(\varepsilon_{i-1}^*, \varepsilon_i^*, y, d_{i-1})/2,$$
$$i = 1,2,\cdots,n \text{ for case 4} \quad (20)$$

TABLE I
SELECTIONS OF SYMBOLS ($m \neq 0$) FOR EIGHT CASES IN FIG. 2

Symbol	Case 1	Case 2	Case3	Case4	Case 5	Case 6	Case 7	Case 8
Σ	Σ'	Σ''	Σ''	Σ'	Σ'	Σ'	Σ''	Σ'
γ_m	$m\pi/a$	$m\pi/2a$	$m\pi/2a$	$m\pi/a$	$m\pi/a$	$m\pi/a$	$m\pi/2a$	$m\pi/a$
A_m	cos	sin	sin	cos	sin	sin	sin	sin
D_m	MK_1	MK_2	MK_1	MK_1	MK_2	MK_1	MK_1	MK_1
$MK_1(n)$	cosh	cosh	cosh	sinh	cosh	cosh	sinh	sinh
$MK_2(n)$	sinh	sinh	sinh	cosh	sinh	sinh	cosh	cosh
$YK_1(1)$	cosh	sinh	cosh	cosh	sinh	cosh	cosh	cosh
$YK_2(1)$	sinh	cosh	sinh	sinh	cosh	sinh	sinh	sinh

$\Sigma' \equiv \sum_{m=1,2,3}^{\infty}$; $\Sigma'' \equiv \sum_{m=1,3,5}^{\infty}$; $MK(n) \equiv MK(\varepsilon_n^*,\varepsilon_{n+1}^*,d_n,d_{n-1},\gamma_m)$;

$YK(1) \equiv YK(\varepsilon_0^*,\varepsilon_1^*,d_1,d_0,\gamma_m)$.

$$MK_3(\varepsilon_i^*,\varepsilon_{i+1}^*,d_i,d_{i-1}) \equiv \varepsilon_i^* MK_3(\varepsilon_{i+1}^*,\varepsilon_{i+2}^*,d_{i+1},d_i) + \varepsilon_{i+1}^*(d_i - d_{i-1}) MK_4(\varepsilon_{i+1}^*,\varepsilon_{i+2}^*), \quad i = 1, 2, \cdots, n-1 \tag{21}$$

$$MK_4(\varepsilon_i^*,\varepsilon_{i+1}^*) \equiv \varepsilon_{i+1}^* MK_4(\varepsilon_{i+1}^*,\varepsilon_{i+2}^*), \quad i = 1, 2, \cdots, n-1 \tag{22}$$

$$YK_3(\varepsilon_i^*,\varepsilon_{i+1}^*) \equiv \varepsilon_i^* YK_3(\varepsilon_{i-1}^*,\varepsilon_i^*), \quad i = 1, 2, \cdots, n-1 \tag{23}$$

$$YK_4(\varepsilon_i^*,\varepsilon_{i+1}^*,d_{i+1},d_i) \equiv \varepsilon_i^*(d_{i+1} - d_i) YK_3(\varepsilon_{i-1}^*,\varepsilon_i^*) + \varepsilon_{i+1}^* YK_4(\varepsilon_{i-1}^*,\varepsilon_i^*,d_i,d_{i-1}), \quad i = 1, 2, \cdots, n-1 \tag{24}$$

$$MK_3(\varepsilon_n^*,\varepsilon_{n+1}^*,d_n,d_{n-1}) \equiv d_n - d_{n-1} \qquad MK_4(\varepsilon_n^*,\varepsilon_{n+1}^*) \equiv 1 \tag{25}$$

$$YK_3(\varepsilon_0^*,\varepsilon_1^*) \equiv 1 \qquad YK_4(\varepsilon_0^*,\varepsilon_1^*,d_1,d_0) \equiv d_1 - d_0 \tag{26}$$

where, the symbols which have two definitions for $m \neq 0$; that is, Σ and γ_m in (4), A_m in (5), D_m in (8), $MK_1(\varepsilon_n^*,\varepsilon_{n+1}^*,d_n,d_{n-1},\gamma_m)$ in (13), $MK_2(\varepsilon_n^*,\varepsilon_{n+1}^*,d_n,d_{n-1},\gamma_m)$ in (14), $YK_1(\varepsilon_0^*,\varepsilon_1^*,d_1,d_0,\gamma_m)$ in (15), and $YK_2(\varepsilon_0^*,\varepsilon_1^*,d_1,d_0,\gamma_m)$ in (16) are selected for the eight cases as shown in Table I, respectively.

Using this general form, we can derive the Green's functions shown by Gish and Graham [2] and by Yamashita and Atsuki [3], [4].

Furthermore, we can easily show that these Green's functions satisfy the following reciprocity relation:

$$G(x_i,y_i;x_j y_j) = G(x_j,y_j;x_i,y_i). \tag{27}$$

The approximate Green's function for a case of open microstrip line can be derived from the Green's function for case 1 by letting both $MK_1(n)$ and $MK_2(n)$ for case 1 in Table I be $\exp(-\gamma_m d_{n-1})$ and letting the sideward dimension a be large. Then, although the parameters for such a case can be obtained by the variational technique [2]–[4] using that Green's function, there is the undesirable property that the infinite series converge slowly when the sideward dimension is large.

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Comments on "Design Equations for an Interdigitated Directional Coupler"

JOSEPH A. MOSKO

In the above short paper,[1] Ou derived some design equations for interdigital couplers. The purpose of this letter is to point out a limitation which may not be apparent upon reading the above work.

Because of Ou's simplifying assumptions, his method is not as accurate as he indicates. Let's consider the very popular −3-dB coupler example he selected. Turning to carefully plotted Bryant & Weiss data by Chambers [1, fig. 2], one can see that although S/H agree quite well, there is an obvious 30-percent disagreement in the W/H values of theory versus experiment [2], [3]. (Actually, Lange's data are *very* impressive —especially because he was announcing a new structure whose dimensions were intuitively and experimentally derived.)

Perhaps some will think that this letter is nit-picking. However, it can be shown by more rigorous theory than Ou used that "significant" strip-width errors can lead to couplers with poor isolation. Because the sources for poor isolation are many (uneven mode velocities in coupled microstrips, tolerances, connectors, wire bonds, etc.), the contributions of a linewidth error may not be obvious. If the primary error of the approximate method were in gap width, this letter would not be necessary because an error in coupling would be more apparent and the fix is obvious.

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Correction to "Analysis of Microstrip-Like Transmission Lines by Nonuniform Discretization of Integral Equations"

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In the above paper,[1] on page 195, the last line of the Abstract, and on page 198, line 4 of Section V, the abbreviation LSM was erroneously said to represent "linear synchronous motor." The correct meaning is "longitudinal-section magnetic modes."

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